JOURNAL OF COMBINATORIAL THEORY, Series A 32, 252–255 (1982)

On the Partial Geometry pg(6, 6, 2)

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Received June 10, 1981

DEDICATED TO PROFESSOR MARSHALL HALL, JR., ON THE OCCASION OF HIS RETIREMENT

We give some new representations of the partial geometry pg(6, 6, 2), which was constructed by van Lint and Schrijver, show a connection with a strongly regular graph based on the ternary Golay code, and determine the automorphism group of the geometry.

1. REPRESENTATION OF pg(6, 6, 2)

In this note we assume that the reader is familiar with the definitions of a *strongly regular graph* $\mathrm{srg}(n, k, \lambda, \mu)$ and of *a partial geometry* $\mathrm{pg}(s+1, t+1, \alpha)$ (cf. [2, pp. 17, 33]). The point graph of a $\mathrm{pg}(s+1, t+1, \alpha)$, where adjacency corresponds to collinearity, is a $\mathrm{srg}(n, k, \lambda, \mu)$ with parameters $n = (s+1)(st+\alpha)/\alpha$, $k = s(t+1)$, $\lambda = s-1+t(\alpha-1)$, $\mu = (t+1)\,\alpha$. With two exceptions, all known constructions of partial geometries lead to infinite sequences. In [4], van Lint and Schrijver constructed the first "sporadic" partial geometry, a pg(6, 6, 2) corresponding to a srg(81, 30, 9, 12). We will give several definitions of this geometry, beginning with the simplest. Let $\bar{C}$ be the ternary repetition code of length 6, i.e.,

$$\bar{C} = \{(0, 0, 0, 0, 0, 0), (1, 1, 1, 1, 1, 1), (2, 2, 2, 2, 2, 2)\}.$$

Any coset of $\bar{C}$ in $\mathbb{F}_3^6$ has a well-defined *type* in $\mathbb{F}_3$, viz., the sum of the coordinates of any vector in the coset. Let $\mathscr{A}_i$ be the set of cosets of type i. We define a tripartite graph Γ by joining the coset $\bar{C} + \mathbf{v}$ to the coset $\bar{C} + \mathbf{v} + \mathbf{w}$

0097-3165/82/020252–04$02.00/0

for each vector $\mathbf{w}$ of weight 1. (Then any vertex in $\mathscr{A}_i$ has six neighbours in $\mathscr{A}_{i+1}$ and six in $\mathscr{A}_{i+2}$.)

Consider the incidence structure with point set $\mathscr{A}_i$ and line set $\mathscr{A}_{i+1}$, in which incidence is defined by adjacency in Γ. This is a pg(6, 6, 2). (For example, suppose $i = 0$, and let p be the point with coset representative $(0, 0, 0, 0, 0, 0)$. The six lines incident with p have representatives of the form $(1, 0, 0, 0, 0, 0)$, and so the 30 points collinear with p have representatives of the form $(1, 2, 0, 0, 0, 0)$. We see that two points are incident with at most one line. A line not incident with p has a representative of the form $(2, 2, 0, 0, 0, 0)$ or $(2, 1, 1, 0, 0, 0)$; in either case, it is incident with two points collinear with p.)

We have in fact constructed three "linked" partial geometries. Specifically, call elements of $\mathscr{A}_0$, $\mathscr{A}_1$ and $\mathscr{A}_2$ *points*, *lines* and *negative lines*, respectively. Both the lines and the negative lines define partial geometries on the point set, and the two partial geometries have the same point graph. A line L and a negative line M meet in at most two points; calling L and M "incident" whenever $|L \cap M| = 1$ defines the third partial geometry.

The set of vectors with coordinate sum 0 can be regarded as being obtained from $\mathbb{F}_3^5$ by adjoining a parity check. So we can identify the set $\mathscr{A}_0$ of points with the set of cosets of the repetition code C of length 5 in $\mathbb{F}_3^5$, and hence with the set of vectors in $C^\perp$. It is already checked that

$$S = \{(0, 0, 0, 0, 0, 0), (2, 1, 1, 1, 1), (1, 2, 1, 1, 1), \ldots, (1, 1, 1, 1, 2)\}$$

is a line. The other lines are its translates $S + \mathbf{a}$ for $\mathbf{a} \in C^\perp$. Two points of $C^\perp$ are collinear if their Hamming distance is 2 or 5. This is the original description given by van Lint and Schrijver [4]. Note that the set $-S$ is a negative line.

The point graph of the partial geometry is a srg(81, 30, 9, 12). It can also be defined as follows. Set $Q(\mathbf{x}) = \sum_{i=1}^5 x_i^2$, a quadratic form on $\mathbb{F}_3^5$. Clearly the value of $Q(\mathbf{x})$ is congruent (mod 3) to the Hamming weight of $\mathbf{x}$. So two vectors $\mathbf{x}$ and $\mathbf{y}$ of $C^\perp$ are collinear if and only if $Q(\mathbf{x} - \mathbf{y}) = -1$.

We require the following facts about the point graph. The non-edges $\{\mathbf{x}, \mathbf{y}\}$ are of two types, which can be distinguished by the structure of the subgraph on the 12 common neighbours of $\mathbf{x}$ and $\mathbf{y}$: this subgraph consists of three copies of $K_{2,2}$ in one case and two copies of $K_{3,3}$ in the other. (This is most easily verified in our first representation.) Furthermore if $\mathbf{x}$ and $\mathbf{y}$ are adjacent, the nine points adjacent to both consist of two copies A, B of K_4 and an isolated vertex $\mathbf{z}$; here $\{\mathbf{x}, \mathbf{y}\} \cup A$ is a line of the partial geometry, $\{\mathbf{x}, \mathbf{y}\} \cup B$ is a negative line, and $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a line of the affine space.

2. Connection with the Golay Code

The ternary Golay code G_{11} is the $[11, 6]$ linear code with generator matrix

$$\left(\begin{array}{ccccc|c|ccccc}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}\right)$$

We remark that $(1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0)$ is a word in G_{11} and $G_{11}^{\perp}$. In [1], Berlekamp, van Lint and Seidel defined a srg(243, 22, 1, 2) by taking the 243 cosets of G_{11} in $\mathbb{F}_3^{11}$ as vertices and joining the coset $G_{11} + \mathbf{v}$ to the coset $G_{11} + \mathbf{v} + \mathbf{w}$ for each vector $\mathbf{w}$ of weight 1. We denote this graph by *BLS*.

Every edge $\{G_{11} + \mathbf{v},\ G_{11} + \mathbf{v} + \mathbf{w}\}$ of *BLS* is in a unique triangle $\{G_{11} + \mathbf{v},\ G_{11} + \mathbf{v} + \mathbf{w},\ G_{11} + \mathbf{v} + 2\mathbf{w}\}$, which is a line of the 5-dimensional affine space $\mathbb{F}_3^{11}/G_{11}$. If we take the triangles of *BLS* as "lines," we obtain a semi-partial geometry, in the sense of Debroey and Thas [3].

The words in G_{11} which end in five 0's form a subcode which is $\bar{C}$ (with five extra 0's). It follows that there is a natural bijection between the cosets of G_{11} in $\mathbb{F}_3^{11}$ and those of $\bar{C}$ in $\mathbb{F}_3^6$. Moreover if two cosets of $\bar{C}$ are at distance 1, then so are the corresponding cosets of G_{11}. So our tripartite graph Γ is a subgraph of *BLS*.

3. The Automorphism Group

Our definition of pg(6, 6, 2) using the cosets of $\bar{C}$ has the advantage that is immediately clear that the group S_6 of coordinate permutations acts on the geometry. Furthermore, both definitions show that the elementary abelian group of order 3^4 acts (by translation) as a group of automorphisms. We claim that these generate the whole group.

Theorem. $\mathrm{Aut}(\mathrm{pg}(6, 6, 2)) = V_{3^4} \cdot S_6$ *(semi-direct product).*

Proof. (i) Our observations in Section 1 on the two kinds of non-edges, applied to the points nonadjacent to $\mathbf{0}$, show that the graph determines the quadratic form Q as function on the vertex set. In fact, for any vectors $\mathbf{x}$ and $\mathbf{y}$, $Q(\mathbf{x} - \mathbf{y})$ is determined. Now the inner product is given by

$$(\mathbf{x}, \mathbf{y}) = Q(\mathbf{x} - \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}).$$

The hyperplanes are the sets $\{\mathbf{x} \mid (\mathbf{x}, \mathbf{v}) = i\}$ for all non-zero vectors $\mathbf{v}$ and all $i \in \mathbb{F}_3$. It follows that the graph determines the affine space.

(*Remark.* It is also possible, though less straightforward, to determine directly the third vertex on the affine line joining any pair of vertices.)

(ii) Since any line through $\mathbf{0}$ contains a basis of $C^{\perp}$, the pointwise stabilizer of such a line must fix everything. It follows that $|\mathrm{Aut}(\mathrm{pg}(6, 6, 2))| \leqslant 3^4 \cdot 6!$, and since we have a group of this size the proof is complete. ∎

Now let us consider the corresponding srg(81, 30, 9, 12). We have seen that every edge is in two 6-cliques, corresponding to a line and a negative line of pg(6, 6, 2). The 6-cliques fall into two orbits of size 81 under automorphisms of pg(6, 6, 2). The transformation $-I$(which maps S into $-S$) interchanges lines and negative lines and induces an automorphism of the graph. It follows that the automorphism group of srg(81, 30, 9, 12) is twice as large as that of pg(6, 6, 2), and in fact is $V_{3^4}.\,(C_2 \times S_6)$.

Similarly, the automorphism group of the tripartite graph Γ is $V_{3^5} \cdot (C_2 \times S_6)$.

Acknowledgments

This paper was born while the authors were returning from a meeting at Oberwolfach; we would like to express our gratitude to participants at the meeting (especially W. Haemers and R. A. Liebler) who contributed valuable comments.

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